LECTURE NOTES

MA 109C: Introduction to Geometry and Topology

Spring 2026

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Preamble

About grades: please don't stress about grades. The lowest score I've ever given is a single A-, and that was in my first year here. From my second year and thereon, I only gave A's and A+'s (and P's for pass/fail options). I am in my third and last year here. You will almost certainly get an A or A+.[1]

I do this because I want you to learn from these courses. I want you to approach the classes with the mindset that you will get something out of them, rather than trying to boost your GPA and add another line to your transcript. I will almost certainly make the GPA/transcript not an issue for you, so that you can focus on learning. Of course, there are those who excel in this class and wish to be distinguished. For those people, I will give an A+; that's why I don't give A+'s to everybody.

Since I want you to learn in this class, please do not use LLMs to blindly solve the homework problems. I use LLMs as well, and they have become very useful: I use them to search the literature, edit my research papers, find and evaluate books in fields outside my expertise, and handle other mundane tasks. However, if I use LLMs to simply solve a problem, I don't learn much from it. Especially with an undergraduate-level class, you have to struggle with the problem from start to finish, digest the material, find moments of inspiration, and make the problem your own. There is great joy in feeling your brain power grow.

This term, I am going to experiment with oral examinations. One reason is that mathematics is not done in a sealed, frictionless vacuum; rather, it is done collaboratively. That means that you need to communicate your mathematical thoughts to others. The exact format of the exam is yet to be determined, but it will be a mathematical conversation between you and me, where I give you a problem, you explain the solution to me, and I ask some clarifying questions about your solution.

Again, because of the collaborativeness of mathematical research, I encourage you to collaborate on problem sets. You should indicate on your solutions with whom you collaborated. This will become a habit of acknowledging people in research papers or even co-authoring with them.

Also, at the risk of sounding pedantic, I encourage you to attend as many of the classes as possible. I won't take attendance, so no pressure, but if you want to learn, please show up. In my experience, those who consistently show up to class tend to perform better. By attending classes, you are pushed to keep up with the material, know what's going on in class, and build intuition for the class material.

[1] Of course, exceptions include but are not limited to: physically attacking me, verbally insulting me, committing crimes against me, or being completely unresponsive (i.e., not submitting any homework/exams while not responding to my emails). But none of these has ever happened in my career so far.

If you feel behind, please don't use that as an excuse not to show up to class, since you will get further behind by not showing up; if you feel completely lost, please don't hesitate to show up to office hours to ask me questions (you can book extra office hours with me if the stated hours don't work for you).

You will have various motivations for taking this class, and you will pursue diverse careers after graduating. Some of you will become research mathematicians, some specializing in geometry, and others in different areas of math. Some of you will research fields other than math, and some of you will take industry jobs. Regardless of your reasons for taking this course, Differential Topology will provide you with a new way of looking at geometry and will enrich your mathematical toolkit. Let us also appreciate that, despite our varied life paths, we are intersecting at this point in space and time.

Let's have fun for the next two and a half months.

— 1 —

Manifolds and Smooth Maps

§ What is differential topology?

Defn. Let $U \subseteq \mathbb{R}^k$, $V \subseteq \mathbb{R}^l$ be open. A map $f: U \to V$ is <u>smooth</u> <u>(or C^∞)</u> if all of its partial derivatives $\frac{\partial^n f}{\partial x_{i_1} \cdots \partial x_{i_n}}$ exist and are continuous.

Defn. Let $X \subseteq \mathbb{R}^k$, $Y \subseteq \mathbb{R}^l$ be arbitrary subsets. A map $f: X \to Y$ is <u>smooth</u> if $\forall x \in X$ $\exists$ open nbhd U of x and $\exists F: U \to \mathbb{R}^l$ smooth s.t. $F = f$ on $U \cap X$.

$f: X \to Y$ is a <u>diffeomorphism</u> if f is bijective and f, f^{-1} are smooth.

Defn. $M \subseteq \mathbb{R}^k$ is a <u>smooth manifold of dimension m</u> if $\forall x \in M$ $\exists$ nbhd W of x in M s.t. W diffeo to an open subset of $\mathbb{R}^m$.

Differential topology studies properties of manifolds that are invariant under diffeomorphism.

It permits us to prove:

- the fundamental theorem of algebra (again)
- the Brouwer fixed point thm
  (every continuous map from the closed unit disk to itself has a fixed point)
- the Jordan-Brouwer separation theorem
  (every smooth curve in $\mathbb{R}^2$ separates its complement into an "inside" and an "outside")
- the Borsuk-Ulam theorem
  (every conti map $f: S^2 \to \mathbb{R}^2$ has a point $p \in S^2$ s.t. $f(-p) = f(p)$)
- that every conti vector field on S^2 vanishes somewhere

and many more.

NO.

§ Tangent Spaces and Derivatives.

Intuitively, the tangent space T_xM of an m-dim mfd $M \subseteq \mathbb{R}^N$ at $x \in M$ is the m-dimensional hyperplane which best approximates M near x. The differential $df_x : T_xM \to T_{f(x)}N$ of a smooth map $f: M \to N$ is the best linear approximation of f near $x \in M$.

- Given open $U \subseteq \mathbb{R}^N$, $x \in U$, define $T_xU := \mathbb{R}^N$.

Given open $U \subseteq \mathbb{R}^N$, $V \subseteq \mathbb{R}^M$, $f: U \to V$ smooth, $x \in U$, define

$$df_x : \mathbb{R}^N \to \mathbb{R}^M,$$

$$df_x(h) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}.$$

This is the linear map given by the matrix $\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{ij}$.

1. Chain rule: if $f: U \to V$, $g: V \to W$ smooth, $f(x) = y$, then

$$d(g \circ f)_x = dg_y \circ df_x.$$

$$\begin{tikzcd} & V \arrow[dr, "g"] & \\ U \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & \circlearrowright & W \end{tikzcd} \Rightarrow \begin{tikzcd} & \mathbb{R}^M \arrow[dr, "dg_y"] & \\ \mathbb{R}^N \arrow[ur, "df_x"] \arrow[rr, "d(g \circ f)_x"'] & \circlearrowright & \mathbb{R}^P \end{tikzcd}$$

2. If $i : U \to U'$ is the natural inclusion, then $di_x = id_{\mathbb{R}^N}$.
3. If $f: U(\overset{\text{open}}{\subseteq} \mathbb{R}^N) \to V(\subseteq \mathbb{R}^M \text{ open})$ diffeo, then $df_x : \mathbb{R}^N \to \mathbb{R}^M$ isom, $N = M$.
4. (Inverse Function Thm) Conversely, if $f: U(\subseteq \mathbb{R}^N \text{ open}) \to V(\subseteq \mathbb{R}^M \text{ open})$ s.t. $df_x : \mathbb{R}^N \to \mathbb{R}^M$, then f is local diffeo at x, i.e. $\exists$ nbhd U' of x s.t. $f|_{U'} : U' \to f(U')$ diffeo.

- Let $M \subseteq \mathbb{R}^N$ be an m-mfd, let $x \in M$.

<u>Def</u> Let $g: U(\subseteq \mathbb{R}^m) \to M$ be a parametrization w/ $g(u) = x$.

The tangent space of M at x is the image of $dg_u : \mathbb{R}^m \to \mathbb{R}^N$.

<u>check!</u> T_xM does not depend on choice of g.

pf)

$$\begin{tikzcd} & \mathbb{R}^N & \\ U \arrow[ur, "g"] \arrow[rr, "h^{-1} \circ g"'] & \circlearrowright & V \arrow[ul, "h"'] \end{tikzcd} \Rightarrow \begin{tikzcd} & \mathbb{R}^N & \\ \mathbb{R}^m \arrow[ur, "dg_u"] \arrow[rr, "\cong"', "d(h^{-1} \circ g)_u"'' below] & \circlearrowright & \mathbb{R}^m \arrow[ul, "dh_v"'] \end{tikzcd}$$

$\mathrm{im}(dg_u) = \mathrm{im}(dh_v)$.

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<u>Check 2</u> T_xM is m-dimensional.

$g^{-1}: g(U) \to U$ is smooth, so extends to $F: W \to \mathbb{R}^m$ smooth. ($W \subset \mathbb{R}^N$ open)

$$\begin{tikzcd} & W \arrow[dr, "F"] & \\ U \arrow[ur, "g"] \arrow[rr, hook, "\text{inclusion}"'] & \circlearrowleft & \mathbb{R}^m \end{tikzcd} \implies \begin{tikzcd} & \mathbb{R}^N \arrow[dr, "dF_x"] & \\ \mathbb{R}^m \arrow[ur, "dg_u"] \arrow[rr, "id"'] & \circlearrowleft & \mathbb{R}^m \end{tikzcd} \quad \therefore dg_u \text{ injective.}$$

cf) See Exercise 1.2.12 for a characterization of T_xM as velocity vectors.

- <u>Def</u> Let $f: M \to N$ be a smooth map. Let $g: U \to M$, $h: V \to N$ be parametrizations with $f(g(U)) \subseteq h(V)$. Then $df_x := dh_v \circ d(h^{-1}\circ f \circ g)_u \circ d(g^{-1})_x$.

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ U \arrow[u, "g"] \arrow[r, "h^{-1}\circ f\circ g"'] \arrow[ur, phantom, "\circlearrowleft"] & V \arrow[u, "h"'] \end{tikzcd} \implies \begin{tikzcd} T_xM \arrow[r, "df_x"] & T_yN \\ \mathbb{R}^m \arrow[u, "dg_u"] \arrow[r, "d(h^{-1}\circ f\circ g)_u"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{R}^n \arrow[u, "dh_v"'] \end{tikzcd}$$

<u>Check 1</u> Well-defined (indep of g, h).

$$\begin{tikzcd} M \arrow[rrr, "f"] & & & N \\ & U \arrow[ul, "g"'] \arrow[r, "h^{-1}\circ f\circ g"] & V \arrow[ur, "h"] & \\ U' \arrow[uu, "g'"] \arrow[ur, "g^{-1}\circ g'"] \arrow[rrr, "h'^{-1}\circ f\circ g'"'] & & & V' \arrow[uu, "h'"'] \arrow[ul, "h^{-1}\circ h'"'] \end{tikzcd} \implies \begin{tikzcd} T_xM \arrow[rrr, "df_x"] & & & T_yN \\ & \mathbb{R}^m \arrow[ul, "\cong"'] \arrow[r] & \mathbb{R}^n \arrow[ur, "\cong"] & \\ \mathbb{R}^m \arrow[uu, "ss"] \arrow[ur, "\cong"] \arrow[rrr] & & & \mathbb{R}^n \arrow[uu, "ss"'] \arrow[ul, "\cong"'] \end{tikzcd}$$

<u>Check 2</u> Chain rule: $d(g\circ f)_x = dg_y \circ df_x$.

$$\begin{tikzcd} M \arrow[rr, "g\circ f"] \arrow[dr, "f"] & & P \\ & N \arrow[ur, "g"'] & \\ & V \arrow[u, "h"] \arrow[dr, "k^{-1}\circ g\circ h"] & \\ U \arrow[uuu, "\bar{g}"] \arrow[ur, "h^{-1}\circ f\circ \bar{g}"] \arrow[rr, "k^{-1}\circ g\circ f\circ \bar{g}"'] & & W \arrow[uuu, "k"'] \end{tikzcd} \implies \begin{tikzcd} T_xM \arrow[rr, "d(g\circ f)_x"] \arrow[dr, "df_x"] & & T_zP \\ & T_yN \arrow[ur, "dg_y"'] & \\ & \mathbb{R}^n \arrow[u] \arrow[dr, "\cong"] & \\ \mathbb{R}^m \arrow[uuu, "ss"] \arrow[ur, "\cong"] \arrow[rr, "\cong"'] & & \mathbb{R}^p \arrow[uuu, "ss"'] \end{tikzcd}$$

- If $i: M \to N$ inclusion map, then $di_x: T_xM \to T_xN$ inclusion map.
- If $f: M \to N$ diffeo, then $df_x: T_xM \to T_{f(x)}N$ isomorphism.
- (Inverse Function Theorem) If $df_x: T_xM \to T_{f(x)}N$ isomorphism, then f local diffeo at $x \in M$.

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§ Regular values.

Def. Let $\dim M = \dim N$, $f: M \to N$ smooth.

$x \in M$ is a <u>regular point</u> of f if df_x iso (f loc diffeo near x).

$x \in M$ is a <u>critical point</u> of f if df_x singular.

$y \in N$ is a <u>regular value</u> of f if $\forall x \in f^{-1}(y)$ regular.

$y \in N$ is a <u>critical value</u> of f if $\exists x \in f^{-1}(y)$ critical.

Thm $\dim M = \dim N$, M cpt, $y \in N$ reg val of $f: M \to N$,

then $\# f^{-1}(y) < \infty$, locally constant as a function of y.

("Stack of records thm", Exercise 1.4.7.)

§ The Fundamental thm of Algebra.

Every nonconstant complex polynomial $P(z)$ has a zero.

Sketch of proof.

Step 1. $P: \mathbb{R}^2 \to \mathbb{R}^2$ defines a map. Its critical values are the images of P', hence there are only finitely many crit. vals.

Step 2. Recall stereographic projection $h_+: S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$. (see Exercise 1.1.12)

Step 3. Define the map $f: S^2 \to S^2$ by

$$f = \begin{cases} h_+^{-1} P h_+ & \text{on } S^2 \setminus \{(0,0,1)\}, \\ (0,0,1) & \text{at } (0,0,1). \end{cases}$$

This is smooth and has finitely many critical values.

$\because h_- f h_-^{-1}(z) = \dfrac{z^n}{\bar{a}_n + \bar{a}_{n-1} z + \cdots + \bar{a}_n z^n}$ if $P(z) = a_n z^n + \cdots + a_0$, use $a_n \neq 0$.

Step 4. The set of regular values of f is connected, so $\# f^{-1}(y)$ is constant for y reg val. If P had no zero, $h_+^{-1}(0)$ is a regular value that is not attained. So no regular value is attained. So $\mathrm{image}(f) =$ crit. vals of f. But $\mathrm{image}(f)$ is connected and finite, hence is a singleton, i.e. $\{(0,0,1)\}$. $\divideontimes$.

§ Immersions.

Def. A map $f: X \to Y$ is an <u>immersion</u> at $x \in X$ if $df_x : T_xX \to T_{f(x)}Y$ is injective.
f is an <u>immersion</u> if it is an immersion at every $x \in X$.

Local Immersion Thm.

Let $f: X \to Y$ be an immersion at $x \in X$, let $y = f(x)$.
There are local coordinates around x and y s.t.

$$f(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0) \text{ in local coordinates.}$$

pf) Choose coordinates so that $df_x = \begin{pmatrix} I_k \\ 0 \end{pmatrix}$, $f: U(\subseteq \mathbb{R}^k) \to V(\subseteq \mathbb{R}^\ell)$, $0 \mapsto 0$.
f is the restriction of $F: U \times W \to V$, $F(x_1, \dots, x_k, z_{k+1}, \dots, z_\ell) = f(x_1, \dots, x_k) + (0, \dots, 0, z_{k+1}, \dots, z_\ell)$.
F is a local diffeo.

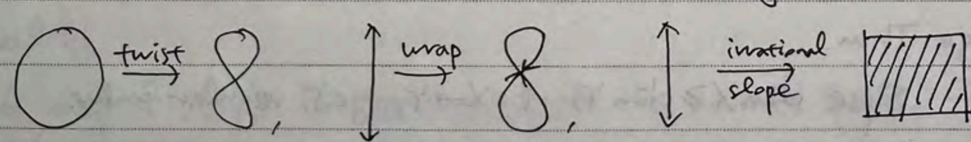
$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "i"'] & V \\ & U \times W \arrow[u, "F"', "\cong"] \end{tikzcd}$$

Cor. If $f: X \to Y$ is an immersion at $x \in X$, it is an immersion on a nbhd of x.

§ Embeddings.

Q. When is an immersion a diffeo onto its image?

Non-Examples



should be injective — should send far away pts to far away pts.

$\Rightarrow$ <u>Def</u> A map $f: X \to Y$ is <u>proper</u> if $\forall$ cpt $K \subseteq Y$, $f^{-1}(K)$ is cpt.

Defn. An embedding $f: X \to Y$ is a diffeomorphism onto its image.

Thm. An injective proper immersion $f: X \to Y$ is an embedding.

pf) Enough to show that f is a homeo onto its image.
Enough to show f closed, i.e., maps closed sets to closed sets.
If $C \subseteq X$ closed, $y \in Y \setminus f(C)$, take nbhd V of y in Y s.t. $\bar{V}$ cpt.
Then $f^{-1}(\bar{V})$ cpt, $C \cap f^{-1}(\bar{V})$ cpt, $f(C \cap f^{-1}(\bar{V})) = f(C) \cap \bar{V}$ cpt, so $V \setminus f(C)$ open nbhd of y.

§ Submersions.

Def. $f: X \to Y$ is a <u>submersion</u> at $x \in X$ if $df_x : T_xX \to T_{f(x)}Y$ is surjective.
$f: X \to Y$ is a <u>submersion</u> if it is a submersion at every $x \in X$.

Local Submersion Thm.

If $f: X^k \to Y^\ell$ is a submersion at $x \in X$, $f(x) = y$, then there are coordinates around x and y so that

$$f(x_1, \cdots, x_k) = (x_1, \cdots, x_\ell) \quad \text{locally.}$$

pf) May choose coordinates so that $df_x = (I_\ell \ 0)$, $f: U \to V$.
Define $F: U \to V \times W$ by $F(x) = (f(x), x_{\ell+1}, \cdots, x_k)$. Then F loc diffeo.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "F"', "\simeq"] \arrow[dr, phantom, "\circlearrowleft"] & V \\ V \times W \arrow[ur, "\pi"'] & \end{tikzcd}$$

§ Preimages.

Def Let $f: X \to Y$. Then $x \in X$ is a <u>regular point</u> if df_x surjective
$x \in X$ is a <u>critical point</u> if df_x not surjective
$y \in Y$ is a <u>regular value</u> if $\forall x \in f^{-1}(y)$ regular pt
$y \in Y$ is a <u>critical value</u> if $\exists x \in f^{-1}(y)$ critical pt.

Preimage Thm.

S'pse $\dim X \geq \dim Y$, $f: X \to Y$, $y \in Y$ regular value.
Then $f^{-1}(y) \subseteq X$ is a submfd of dimension $\dim X - \dim Y$,
and $T_x f^{-1}(y) = \ker(df_x : T_xX \to T_yY)$.

pf) By local submersion thm & matching dimensions.

ex1) $S^{n-1} \subseteq \mathbb{R}^n$ is an $(n-1)$-mfd, since it is preimage of
$\mathbb{R}^n \to \mathbb{R}$, $(x_1, \cdots, x_n) \mapsto x_1^2 + \cdots + x_n^2$, 1 is reg val.
$T_x S^{n-1} = \{y \mid x \cdot y = 0\}$.

ex2) $O(n) = \{A \in \mathbb{R}^{n\times n} \mid AA^t = I_n\}$ is an $\frac{n(n-1)}{2}$-mfd, since it is preimage of
$\mathbb{R}^{n\times n} \to \mathrm{Sym}(\mathbb{R}^{n\times n})$, $A \mapsto AA^t$, I_n is reg val.
$T_{I_n} O(n) = \{B \mid B + B^t = 0\}$.

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NO.

§ Cutting out by independent functions.

Zero sets of functions, also known as varieties in some cases, is an object of study.

Def. Functions $g_1, \ldots, g_\ell : X \to \mathbb{R}$ are <u>independent</u> at $x \in X$ if $d(g_1)_x, \ldots, d(g_\ell)_x : T_xX \to \mathbb{R}$ are linearly independent in the dual space $(T_xX)^*$.

($\Leftrightarrow$ $G := (g_1, \ldots, g_\ell) : X \to \mathbb{R}^\ell$ has regular point $x \in X$).

Preimage Thm.

If $g_1, \ldots, g_\ell : X \to \mathbb{R}$ are independent whenever they all vanish, then $\bigcap_{i=1}^{\ell} g_i^{-1}(0)$ is a submfd of X of codim ℓ.

And $T_x(\bigcap_i g_i^{-1}(0)) = \bigcap_i \operatorname{Ker} d(g_i)_x$.

Partial Converse 1.

If $f : X \to Y$ has reg val y, then $f^{-1}(y)$ can <u>locally</u> be cut out by indep fcns.

(necessary; consider $id : S^1 \to S^1$.
typo on p. 24 of textbook.)

Partial Converse 2.

A submfd of X is locally cut out by indep fcns.

§ Transversality.

Def. · A map $f : X \to Y$ is <u>transversal</u> to the submfd $Z \subseteq Y$, $f \pitchfork Z$, if $\operatorname{im}(df_x) + T_yZ = T_yY$ for any $x \in f^{-1}(Z)$, $y = f(x)$.

· Two submfds $X, Z \subseteq Y$ are <u>transversal</u>, $X \pitchfork Z$, if $T_pX + T_pZ = T_pY$ for any $p \in X \cap Z$.

Thm. If $f : X \to Y$ is transversal to $Z \subseteq Y$, then $f^{-1}(Z) \subseteq X$ is a submfd w/ $\operatorname{Codim}_X f^{-1}(Z) = \operatorname{codim}_Y Z$, $T_x f^{-1}(Z) = df_x^{-1}(T_yZ)$.

pf) locally Z is given as the zero set of a function $g : Y \to \mathbb{R}^\ell$, 0 reg val of g. Then $g \circ f : X \to \mathbb{R}^\ell$ has reg val 0 and $(g \circ f)^{-1}(0) = f^{-1}(Z)$ locally. Also $T_x f^{-1}(Z) = T_x (g \circ f)^{-1}(0) = \operatorname{Ker} d(g \circ f)_x = df_x^{-1}(T_yZ)$.

NO.

Thm If $X \pitchfork Z$ in Y then $X \cap Z$ is a mfd, $\mathrm{codim}_Y(X\cap Z) = \mathrm{codim}_Y X + \mathrm{codim}_Y Z$.
$T_p(X\cap Z) = T_pX \cap T_pZ$.

Rmk If $\dim X + \dim Z < \dim Y$ then $X \pitchfork Z \iff X \cap Z = \phi$.

§ Homotopy and Stability.

Def. Two maps $f_0, f_1 : X \to Y$ are (smoothly) homotopic if there is a homotopy between them, i.e., a map $F: X\times[0,1] \to Y$ s.t.

$$F(x,0) = f_0(x), \quad F(x,1) = f_1(x), \quad x \in X.$$

- This is an equivalence relation. Equivalence classes are homotopy classes.

Def A property of maps is stable if whenever $f_0 : X \to Y$ has that property and F is a homotopy with $F(\cdot,0) = f_0$, then for some $\varepsilon > 0$ $\forall t < \varepsilon$ $f_t = F(\cdot, t)$ has that property.

Stability thm.

The following classes of maps of a cpt mfd X into a mfd Y are stable:
- local diffeos, - immersions, submersions,
$f \pitchfork Z$ for a fixed $Z \subseteq Y$, embeddings diffeos.

pf) First three b/c determinant of minor is continuous
("full rank is open condition").
$f \pitchfork Z$ is locally a submersion condition.
for embeddings, need to show injectivity is preserved.

§ Sard's thm.

If $f: X \to Y$ is a smooth map, then almost every point of Y is a reg. val. of f
($\iff$ the critical vals of f have measure zero in Y.)
$\Rightarrow$ the reg. vals of f are dense in Y.

NO.

What does measure zero mean?

Def. $A \subseteq \mathbb{R}^\ell$ has <u>measure</u> <u>zero</u> if $\forall \varepsilon > 0$ $\exists$ covering $A \subseteq \bigcup_{i=1}^{\infty} Q_i$ by rectangular solids s.t. $\sum \mathrm{vol}(Q_i) < \varepsilon$.

- This is a diffeomorphism invariant. In fact, if $g: U(\subseteq \mathbb{R}^\ell) \to \mathbb{R}^\ell$ is smooth and $A \subseteq \mathbb{R}^\ell$ has meas. zero, so does $g(U \cap A)$.
- A countable union $\bigcup_{i=1}^{\infty} A_i$ of measure zero sets A_i has measure zero.

  pf) $A_i \subseteq \bigcup_{j=1}^{\infty} Q_{i,j}$, $\sum_j \mathrm{vol}(Q_{i,j}) < \frac{\varepsilon}{2^i}$ $\to$ $\bigcup_i A_i \subseteq \bigcup_{i,j} Q_{i,j}$, $\sum_{i,j} \mathrm{vol}(Q_{i,j}) < \varepsilon$.

Def. $C \subseteq M$ has <u>measure</u> <u>zero</u> if for every parametrization $\psi: U(\subseteq \mathbb{R}^m) \to M$, $\psi^{-1}(C)$ has measure zero in $\mathbb{R}^m$.

rmk) Because M is Lindelöf (every cover has a countable subcover), enough to find a covering of M with such params.

Note that nonempty open sets do not have measure zero. So the critical values of an $f: M \to N$ are nowhere dense.

One application of Sard's thm is that some homotopy groups of spheres are trivial.

Thm. Let $i < n$ be positive integers. Any continuous map $f: S^i \to S^n$ is continuously homotopic to a constant map.

pf) <u>Claim</u> Every smooth map $g: S^i \to S^n$ is smoothly homotopic to a constant map.

pf) By Sard's thm, g has a regular value, say $p \in S^n$. This means $p \notin g(S^i)$, so that $g: S^i \to S^n \setminus \{p\}$. But $S^n \setminus \{p\} \cong \mathbb{R}^n$ is contractible. $\square$

By Stone-Weierstrass, there is a polynomial function $\tilde{g}: S^i \to \mathbb{R}^{n+1}$ s.t. $|f - \tilde{g}| < \frac{1}{100}$. Then $g := \frac{\tilde{g}}{|\tilde{g}|}: S^i \to S^n$ satisfies $|f - g| < \frac{1}{50}$, so f and g are htpc via $\frac{(1-t)f + tg}{|(1-t)f + tg|}$.

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§ Proof of Sard's thm

Passing to coordinates, it is enough to prove it when $f: U(\subseteq \mathbb{R}^n) \to V(\subseteq \mathbb{R}^m)$ is a map between Euclidean spaces.

Step 0. The case $n<m$.

Write $U = \bigcup_{i=1}^{\infty} K_i$, K_i cpt, $K_i \subseteq \text{int } K_{i+1}$. For small $\delta>0$, cover K_i by disjoint cubes Q of sidelength δ contained in K_{i+1}. It takes $\frac{C}{\delta^n}$ cubes to do this. Since f is Lipschitz on K_{i+1}, say with constant L, each Q is mapped into a ball of radius $\delta\sqrt{n}L$, and a cube of sidelength $2\delta\sqrt{n}L$. So we have a covering of $f(K_i)$ by $\frac{C}{\delta^n}$ cubes of sidelength $2\delta\sqrt{n}L$, whose volumes sum to $\frac{C}{\delta^n}\cdot(2\delta\sqrt{n}L)^m = C(2\sqrt{n}L)^m\delta^{m-n} \to 0$ as $\delta\to 0$. So $f(K_i)$ has measure zero, so does $f(U) = \bigcup_i f(K_i)$. □

- This is the base case for an induction on dimension.

Let $f: U(\subseteq\mathbb{R}^n) \to V(\subseteq \mathbb{R}^m)$, and denote

$C = \{\text{critical points of } f\}$

$C_i = \{x\in U \mid \text{derivatives of order} \leq i \text{ vanish at } x\in U\}$.

Step 1. $f(C\setminus C_1)$ has measure zero.

Enough to find an open set U° around each point $x_0 \in C\setminus C_1$ s.t. $f(U^\circ\cap C)$ has measure zero. For $x_0\in C\setminus C_1$, up to a change in coordinates we have $\frac{\partial f_1}{\partial x_1}(x_0) \neq 0$. So the map $h: U\to\mathbb{R}^n$,

$$h(x_1,\dots,x_n) = (f_1(x), x_2, \dots, x_n)$$

is locally invertible at x_0, say $h: U^\circ \xrightarrow{\cong} V^\circ$. Also $f|_{U^\circ}$ and $g := f\circ h^{-1}$ have the same critical values. But

$$g(x_1', x_2', \dots, x_n') = (x_1', \overline{g}_{x_1'}(x_2', \dots, x_n')), \quad \left(\frac{\partial g_i}{\partial x_j'}\right) = \begin{pmatrix} 1 & 0 \\ * & \frac{\partial \overline{g}_{x_1'}}{\partial x_j'} \end{pmatrix}$$

If $m=1$, $g(x_1',\dots,x_n') = x_1'$ has no critical values.

If $m=1$, then $(x_1',\dots,x_n')$ is a critical pt of g iff $(x_2',\dots,x_n')$ is a crit pt of $\overline{g}_{x_1'}$. So on the slice $\{x_1'\}\times\mathbb{R}^{n-1}$, critical values of $\overline{g}_{x_1'}$, hence crit. vals of g, have meas 0. By Fubini, the crit vals of g have meas 0.

Step 2. $f(C_i \setminus C_{i+1})$ has measure zero.

pf) Let $x_0 \in C_i \setminus C_{i+1}$. Some partial derivative, say $\frac{\partial^i f}{\partial x_{j_1} \cdots \partial x_{j_i}} =: w$,

is such that $w(x_0) = 0$, $\frac{\partial w}{\partial x_1}(x_0) \neq 0$. Then

$$h : U^\circ \to V^\circ,\ (x_1, \dots, x_n) \mapsto (w, x_2, \dots, x_n)$$

is locally inverteible, and if we define $g := f \circ h^{-1}$, and C_n^g for g,

then $f|_{U^\circ}(C_i \setminus C_{i+1}) = g(C_i^g \setminus C_{i+1}^g)$. Since $h(C_i) \subseteq \{0\} \times \mathbb{R}^{n-1}$,

$C_i^g \setminus C_{i+1}^g \subseteq 0 \times \{\text{crit pts of } \bar{g}\}$, where $\bar{g}(x_2, \dots, x_n) = g(0, x_2, \dots, x_n)$. So

$g(C_i^g \setminus C_{i+1}^g) \subseteq \bar{g}\{\text{crit pts of } \bar{g}\}$ has meas 0.

Step 3. $f(C_i)$ has measure zero if $i > \frac{n}{m} - 1$.

pf) Let $K \subseteq \text{int } K' \subseteq K' \subseteq U$, K and K' cpt. By Taylor expansion,

for a cube Q contained in K', if $x_0 \in Q \cap C_i$, then

$$f(x) = f(x_0) + O(|x - x_0|^{i+1}) \quad \text{if } x \in Q.$$

$$\to |f(x) - f(x_0)| = O(|x - x_0|^{i+1}) \quad \text{if } x \in Q.$$

So if Q is of sidelength δ, ~~then~~ and $Q \cap C_i \neq \phi$, then $f(Q)$ is contained in a ball of radius $O((\sqrt{n}\,\delta)^{i+1})$, has volume $O(\delta^{(i+1)m})$.

There are at most $O(\delta^{-n})$ such balls, so may cover image with volume $\leq O(\delta^{(i+1)m - n}) \to 0$ as $\delta \to 0$.

§ Morse functions.

Def. A critical point p of $f : U (\subseteq \mathbb{R}^n) \to \mathbb{R}$ is <u>nondegenerate</u> if its Hessian

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) = H$$

is nonsingular at p.

<u>Obs</u> At a nondegenerate crit pt p, the map $q \mapsto \left(\frac{\partial f}{\partial x_1}(q), \dots, \frac{\partial f}{\partial x_n}(q) \right)$ is a local diffeo. so nondegenerate critical points are isolated.

<u>Morse Lemma</u> At a nondegenerate critical pt 0 of f, there are local coordinates $y_1, \dots, y_n$ s.t. $f(y_1, \dots, y_n) = \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(0)\, y_i y_j + f(0)$.

Nondegeneracy is invariant under diffeos.

Lem. $\psi : U(\ni 0) \to U'(\ni 0)$ diffeo, $f : U' \to \mathbb{R}$ has 0 as nondegen. crit. pt

$\Rightarrow$ $f \circ \psi$ has 0 as nondegen. crit. pt.

pf) $\frac{\partial (f\circ\psi)}{\partial x_i}(0) = \sum_j \frac{\partial f}{\partial y_j}(0) \frac{\partial \psi_j}{\partial x_i}(0) = 0 \Rightarrow 0$ is crit pt of $f\circ\psi$

$$\frac{\partial^2 (f\circ\psi)}{\partial x_i \partial x_j}(0) = \sum_{k,l} \frac{\partial^2 f}{\partial y_k \partial y_l}(0) \frac{\partial y_k}{\partial x_i}(0) \frac{\partial y_l}{\partial x_j}(0) + \cancel{\sum_k \frac{\partial f}{\partial y_k}(0) \frac{\partial^2 y_k}{\partial x_i \partial x_j}(0)}$$

$$\Rightarrow \left(\frac{\partial^2 (f\circ\psi)}{\partial x_i \partial x_j}(0) \right)_{i,j} = \left(\frac{\partial y_k}{\partial x_i}(0) \right)^t_{i,k} \cdot \left(\frac{\partial^2 f}{\partial y_k \partial y_l}(0) \right)_{k,l} \cdot \left(\frac{\partial y_l}{\partial x_j}(0) \right)_{j,l}. \quad \text{inv.}$$

$\square$

§ proof of Morse lemma.

Up to coordinate change, may assume $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0) \right)_{i,j} = \begin{pmatrix} -I_m & 0 \\ 0 & I_p \end{pmatrix}$.

We have

$$\begin{aligned} f(x_1, \dots, x_n) - f(0) &= \int_0^1 \frac{d}{dt} f(tx)\, dt \\ &= \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)\, dt \\ &= \sum_{i=1}^n x_i \int_0^1 \int_0^1 \frac{d}{ds} \frac{\partial f}{\partial x_i}(tsx)\, ds\, dt \\ &= \sum_{i,j=1}^n x_i x_j \underbrace{\int_0^1 \int_0^1 t \frac{\partial^2 f}{\partial x_i \partial x_j}(tsx)\, ds\, dt}_{=: \frac{1}{2}\tilde{h}_{ij}(x)}. \end{aligned}$$

Then $\tilde{h}_{ij}$ is smooth and $\tilde{h}_{ij}(0) = h_{ij}(0)$.

<u>Fact.</u> There is a neighborhood U of $\begin{pmatrix} -I_m & 0 \\ 0 & I_p \end{pmatrix}$ in $\mathbb{R}^{n\times n}$ s.t.

there is a function $B : U \to \mathbb{R}^{n\times n}$ s.t.

$$B\begin{pmatrix} -I_m & \\ & I_p \end{pmatrix} = I_n, \qquad A = B^t(A) \cdot \begin{pmatrix} -I_m & \\ & I_p \end{pmatrix} \cdot B(A),\ A \in U.$$

So around $x = 0$, we have, denoting $B(x) = B\left((\tilde{h}_{ij}(x))_{ij} \right)$,

$$\begin{aligned} f(x) - f(0) &= \frac{1}{2} x^t \left(\tilde{h}_{ij}(x) \right)_{ij} x = \frac{1}{2} x^t B^t \begin{pmatrix} -I_m & \\ & I_p \end{pmatrix} B x. \\ &= \frac{1}{2} y^t \begin{pmatrix} -I_m & \\ & I_p \end{pmatrix} y, \quad y := Bx. \end{aligned}$$

y is the new coordinate system.

§ Morse functions are "generic"

Lemma. Let $f: U(\subseteq \mathbb{R}^n) \to \mathbb{R}$ be smooth. For almost any $a \in \mathbb{R}^n$, the function $f_a: U \to \mathbb{R}$, $f_a(x) = f(x) + a \cdot x$ has only nondegenerate critical points.

pf) Define $g: U \to \mathbb{R}^n$, $g(x) = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)(x)$.

By Sard's thm, almost every $a \in \mathbb{R}^n$ is such that $-a$ is a regular value of g. But $df_a = g + a$, $\mathrm{Hess}(f_a) = dg$, so for such a, f_a only has nondegen. crit. pts.

Def. For a function $f: X \to \mathbb{R}$, a critical pt $p \in X$ is <u>nondegenerate</u> if it is so in some coord system. f is a <u>Morse function</u> if all of its crit. pts are nondegenerate.

Thm. Define, for $f: X(\subseteq \mathbb{R}^N) \to \mathbb{R}$, the function $f_a: X \to \mathbb{R}$, $a \in \mathbb{R}^N$, by

$$f_a(x) = f(x) + a \cdot x, \quad x \in X.$$

For almost every $a \in \mathbb{R}^N$, f_a is Morse.

pf) Apply Lemma on coordinate patches, use Fubini.

§8. Embedding Manifolds in Euclidean Space.

Def. Given k-mfd X, define <u>tangent bundle</u>

$$T(X) := \{(x,v) \in X \times \mathbb{R}^N : v \in T_xX\}.$$

It is a $2k$-mfd. $X \hookrightarrow X_0 = X \times \{0\}$, $(\{x\} \times \mathbb{R}^N) \cap T(X) = \{x\} \times T_xX$, $T(U) = U \times \mathbb{R}^N$. $U \subseteq \mathbb{R}^N$ open

Def. $f: X \to Y$ induces <u>derivative map</u> $df: T(X) \to T(Y)$,

$$df(x,v) = (f(x), df_x(v)).$$

Chain rule: $d(g \circ f) = dg \circ df$.

Thm. any subset $X \subseteq \mathbb{R}^N$ with open cover $\{U_\alpha\}$ has a subordinate <u>partition of unity</u> $\{\theta_i\}$

- $0 \leq \theta_i \leq 1$, smooth on X
- $\forall x \in X$ $\exists x \in U_x$ all but finitely many θ_i's vanish on U_x
- $\forall i$ $\exists \alpha$ $\operatorname{spt} \theta_i \subseteq U_\alpha$
- $\sum_i \theta_i = 1$

pf) Compact exhaustion,

Cor. $\forall$ closed X $\exists$ proper map $X \to \mathbb{R}$. pf) $\rho = \sum i\theta_i$

Whitney embedding thm. Any k-mfd embeds in $\mathbb{R}^{2k+1}$.

rmk) may improve to $\mathbb{R}^{2k}$. not further: S^1, Klein bottle.

pf) Given injective immersion $f: X \to \mathbb{R}^N$, $N \geq 2k+2$, the image of

$TX \to \mathbb{R}^N$, $(x,v) \mapsto df_x(v)$, $\quad X \times X \times \mathbb{R} \to \mathbb{R}^N$, $(x,y,t) \mapsto t(f(x) - f(y))$

have meas 0, so take a outside these. Project using a.

So have inj imm $X \to \mathbb{R}^{2k+1}$. Done if X cpt. If X noncpt, have inj imm $X \xrightarrow{f} B^{2k+1}$, then use $(f, \rho): X \to B^{2k+1} \times \mathbb{R}$, ρ proper. Project again.

Exercises.

6. vector field (smooth $v: X \to \mathbb{R}^N$, $v(x) \in T_xX$) = cross section (smooth $v: X \to T(X)$, $\pi \circ v = id_X$)
7. $\exists$ vf on S^k (k odd) having no zeroes (i.e. $v \neq 0$)
8. $\exists$ non vanishing vf on S^k implies $-id_{S^k} \sim id_{S^k}$.
9. $f: X \to Y$, $Z \subseteq X$, df iso on Z, $Z \xrightarrow[\cong]{f} f(Z) \Rightarrow \exists_{Z \subset U} U \xrightarrow[\cong]{f} {}_{f(Z) \subset} V$.

"The self-intersections of a smooth n-manifold in 2n-space." Whitney. Annals of Math 1944

"The singularities of a smooth n-manifold in (2n-1)-space." Whitney. Annals of Math 1944